

Veech groups of flat structures on Riemann surfaces

Yoshihiko Shinomiya

ABSTRACT. In this paper, we construct new examples of Veech groups by extending Schmithüsen's method for calculating Veech groups of origamis to Veech groups of unramified finite coverings of regular $2n$ -gons. We calculate the Veech groups of certain Abelian coverings of regular $2n$ -gons by using an algebraic method.

1. Introduction

The Teichmüller disk is a holomorphic isometric embedding of an upper-half plane \mathbb{H} (or a unit disk) into a Teichmüller space. All such embeddings are constructed by flat structures on Riemann surfaces and $\mathrm{SL}(2, \mathbb{R})$ -orbit on flat structures. To study the image of a Teichmüller disk into the moduli space, we consider the stabilizer of the Teichmüller disk in the mapping class group. Veech [Vee89] showed that this stabilizer is regarded as the group of all affine diffeomorphisms on a corresponding flat structure and its action can be represented by a Fuchsian group which acts on \mathbb{H} . The Fuchsian group is called a Veech group.

The first non-trivial examples of Veech groups were given by Veech [Vee89] and [Vee91]. His examples are constructed by gluing two congruent regular polygons along one side and identifying the parallel sides of the resulting polygons. However, not so many examples are known other than Veech's. Recently, Schmithüsen [Sch04] showed an algorithm for finding Veech groups of "origami". An origami is an unramified finite covering of a once punctured torus constructed by a unit square. We apply her method to unramified finite coverings of regular $2n$ -gons instead of the unit square to obtain other examples of Veech groups. Veech groups of universal coverings play an important role in her method. We call these groups universal Veech groups.

In this paper, we determine the universal Veech groups of $2n$ -gons and give an algorithm to calculate Veech groups of finite Abelian coverings of $2n$ -gons. In the case of origamis, Schmithüsen connected the Veech groups of origamis with subgroups of $\mathrm{SL}(2, \mathbb{Z})$. She showed that the calculations of Veech groups stop in finitely many steps. In our case, for the Veech groups of Abelian coverings of $2n$ -gons whose degree is d , we connect them with subgroups of $\mathrm{SL}(n, \mathbb{Z}_d)$. We show that the calculations of Veech groups of certain Abelian coverings can be done by using the corresponding subgroups of $\mathrm{SL}(n, \mathbb{Z}_d)$.

2. Definitions

Let X be a Riemann surface of type (g, n) with $3g - 3 + n > 0$.

Definition 2.1 (Holomorphic quadratic differential). A holomorphic quadratic differential φ on X is a tensor whose restriction to every coordinate neighborhood (U, z) is the form $f dz^2$, here f is a holomorphic function on U .

We define $|\varphi|$ to be the differential 2-form on X whose restriction to every coordinate neighborhood (U, z) has the form $|f| dx dy$ if φ equals $f dz^2$ in U . We say φ is integrable if its norm

$$||\varphi|| = \iint_X |\varphi|$$

is finite.

We fix an integrable holomorphic quadratic differential φ . Denote by X' the Riemann surface constructed from X by removing zeros of φ .

Definition 2.2 (Flat structure). A flat structure u on X' is an atlas of X' which satisfies the following conditions.

- (1) Local coordinates of u are compatible with the orientation on X' induced by its Riemann surface structure.
- (2) For coordinate neighborhoods (U, z) and (V, w) of u with $U \cap V \neq \emptyset$, the transition function is the form

$$w = \pm z + c$$

in $z(U \cap V)$ for some $c \in \mathbb{C}$.

- (3) u is maximal with respect to (1) and (2).

The holomorphic quadratic differential φ determines a flat structure u_φ on X' as follows.

For each $p_0 \in X'$, we can choose an open neighborhood U such that

$$z(p) = \int_{p_0}^p \sqrt{\varphi}$$

is a well-defined and injective function of U . This function is holomorphic in U since φ is a holomorphic quadratic differential. If (U, z) and (V, w) are pairs of such neighborhoods and functions with $U \cap V \neq \emptyset$, then we have $dw^2 = \varphi = dz^2$ in $U \cap V$. Hence $w = \pm z + c$ in $z(U \cap V)$ for some $c \in \mathbb{C}$. The flat structure u_φ is the maximal flat structure which contains such pairs.

Definition 2.3 (Affine group of φ). The affine group $Aff^+(X, \varphi)$ of the integrable holomorphic quadratic differential φ is the group of all quasiconformal mappings f of X onto itself which satisfy $f(X') = X'$ and are affine with respect to the flat structure u_φ . This means that for $(U, z), (V, w) \in u_\varphi$ with $f(U) \subseteq V$, the homeomorphism $w \circ f \circ z^{-1}$ is the form $z \mapsto Az + c$ for some $A \in \text{GL}(2, \mathbb{R})$ and $c \in \mathbb{C}$.

This A is uniquely determined up to the sign since u_φ is a flat structure. And A is always in $\text{SL}(2, \mathbb{R})$ since $||\varphi|| = \int_X |\varphi| = \int_X f^*(|\varphi|) = \det(A) ||\varphi||$. Thus we have a group homomorphism

$$D : Aff^+(X, \varphi) \rightarrow \text{PSL}(2, \mathbb{R}).$$

Definition 2.4 (Veech group of φ). We call $\Gamma(X, \varphi) = D(Aff^+(X, \varphi))$ the Veech group of φ .

Remark. Veech groups are discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$ (see [EG97]).

3. Examples of Veech groups

In this section, we see two examples of Veech groups. The first example is a new example of Veech groups. The second one is the main target of this paper. The purpose of this paper is to determine Veech groups of some coverings of the second one. To do this, we need to determine the Veech group of the second one.

Example 3.1. Let X be a surface constructed as Figure 1. We induce an unique conformal structure on X such that the quadratic differential dz^2 on the interior of the rectangle of Figure 1 extends to a holomorphic quadratic differential φ on X . Then X is a Riemann surface of type $(2, 0)$ and vertices of four squares become two points on X . These points are zeros of φ of order 2. We can see that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ define elements in $Aff^+(X, \varphi)$ as Figure 2. Hence $\Gamma = \left\langle \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right] \right\rangle$ is a subgroup of the Veech group $\Gamma(X, \varphi)$. Since every element in $Aff^+(X, \varphi)$ must preserve the set of all lattice points, $\Gamma(X, \varphi)$ is a subgroup of $\mathrm{PSL}(2, \mathbb{Z})$. It is known that $\left\langle \left[\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right], \left[\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right] \right\rangle$ is the congruence subgroup of level 2 and has index 6 in $\mathrm{PSL}(2, \mathbb{Z})$. Hence $\Gamma(X, \varphi)$ is either Γ or $\mathrm{PSL}(2, \mathbb{Z})$. However, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ cannot be an element in $\Gamma(X, \varphi)$. Therefore $\Gamma(X, \varphi)$ must be Γ .

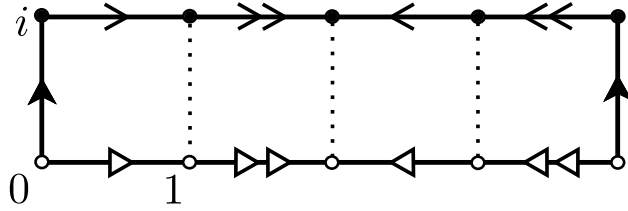


FIGURE 1.

The next example is given by Earle and Gardiner ([EG97]).

Example 3.2. Fix $n \geq 4$ and let Π_{2n} be a regular $2n$ -gon. We assume that Π_{2n} has two horizontal sides, lengths of the sides are 1 and its vertices are removed. We identify each side of Π_{2n} with the opposite parallel side by an Euclidean translation (see Figure 3) and denote the resulting surface by P_{2n} . We induce an unique conformal structure on P_{2n} such that the quadratic differential dz^2 on the interior of Π_{2n} extends to a holomorphic quadratic differential φ_{2n} on P_{2n} . If n is even, then P_{2n} is a Riemann surface of type $(\frac{n}{2}, 1)$ and if n is odd, then P_{2n} is a Riemann surface of type $(\frac{n-1}{2}, 2)$. Now $R_{2n} = \begin{pmatrix} \cos \frac{\pi}{n} & -\sin \frac{\pi}{n} \\ \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}$ and $T_{2n} = \begin{pmatrix} 1 & 2 \cot \frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}$

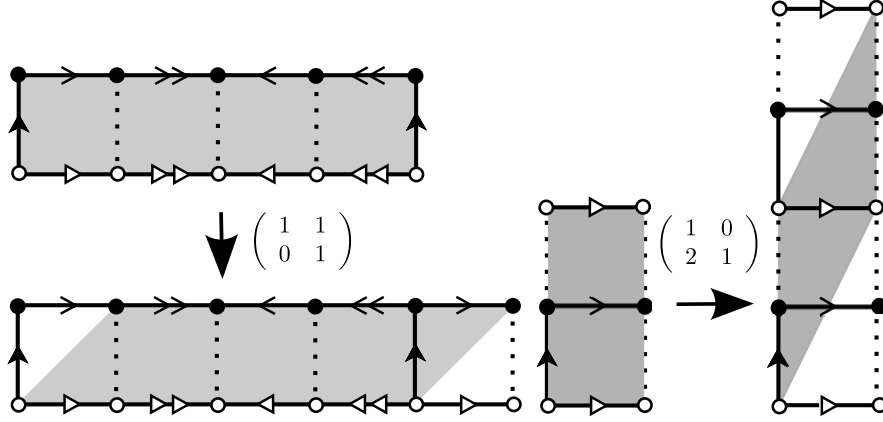


FIGURE 2.

induce elements in $Aff^+(P_{2n}, \varphi_{2n})$. The action of R_{2n} on P_{2n} is the rotation about the center of Π_{2n} of angle $\frac{\pi}{n}$. To see the action of T_{2n} on P_{2n} , we cut P_{2n} along all horizontal segments which connect the vertices of Π_{2n} . If n is even, P_{2n} is decomposed into $\frac{n}{2}$ cylinders and the action of T_{2n} is the composition of the square of the right Dehn twist along a core curve of the cylinder which contains the center of Π_{2n} and the right Dehn twists along core curves of the other cylinders. If n is odd, P_{2n} is decomposed into $\frac{n-1}{2}$ cylinders and the action of T_{2n} is the composition of the right Dehn twists along core curves of all cylinders. Thus $\Gamma = \langle [R_{2n}], [T_{2n}] \rangle$ is a subgroup of the Veech group $\Gamma(P_{2n}, \varphi_{2n})$. It is easy to see that Γ is a (n, ∞, ∞) triangle group. Since only discrete group that contains Γ is a $(2, 2n, \infty)$ triangle group (see [EG97] and [Sin72]) and this cannot be $\Gamma(P_{2n}, \varphi_{2n})$, we have $\Gamma(P_{2n}, \varphi_{2n}) = \langle [R_{2n}], [T_{2n}] \rangle$.

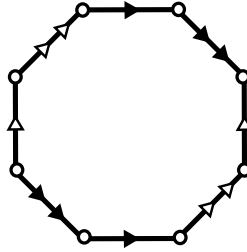


FIGURE 3.

4. Veech groups of coverings of P_{2n} and Universal Veech group of P_{2n}

Fix $n \geq 4$. Let P_{2n} be the same Riemann surface as in Example 3.2 and $p : X \rightarrow P_{2n}$ be an unramified finite covering mapping. Set $\varphi_X = p^* \varphi_{2n}$, here φ_{2n} is the holomorphic quadratic differential on P_{2n} defined in Example 3.2. Our purpose is to calculate the Veech group $\Gamma(X, \varphi_X)$. We denote $\Gamma(X, \varphi_X)$ by $\Gamma(X)$

hereafter. Schmithüsen constructed an algorithm for calculating Veech groups of origamis ([Sch04]). We apply her method to our case.

Let $p_{2n} : \tilde{X}_{2n} \rightarrow P_{2n}$ be the universal covering mapping and set $\tilde{\varphi}_{2n} = p_{2n}^* \varphi_{2n}$. Note that $\|\tilde{\varphi}_{2n}\| = +\infty$. However, we can define the flat structure $u_{\tilde{\varphi}_{2n}}$ on \tilde{X}_{2n} and the affine group $Aff^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ in the same manner as the case of integrable holomorphic quadratic differentials. Moreover, we have a homomorphism $D : Aff^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \rightarrow \text{PGL}(2, \mathbb{R})$. Set $\Gamma(\tilde{X}_{2n}) = \text{Im}(D) \cap \text{PSL}(2, \mathbb{R})$.

Definition 4.1 (Universal Veech group of P_{2n}). We call $\Gamma(\tilde{X}_{2n})$ the universal Veech group of P_{2n} .

Remark. Let X be an unramified finite covering of P_{2n} . Then for each $f \in Aff^+(X, \varphi_X)$, there exists a lift $\tilde{f} \in Aff^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ with $D(\tilde{f}) = D(f)$. Hence $\Gamma(X)$ is a subgroup of $\Gamma(\tilde{X}_{2n})$.

The following idea is due to Schmithüsen ([Sch04]). For each finite covering X of P_{2n} , we take $\Gamma(X)$ as follows.

$$\begin{aligned} \Gamma(X) &= \left\{ [A] \in \Gamma(\tilde{X}_{2n}) \mid \exists \tilde{f} \in Aff^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \text{ s.t. } D(\tilde{f}) = [A], \tilde{f} \text{ is a lift of a} \right. \\ &\quad \left. \text{homeomorphism of } X \text{ onto itself} \right\} \\ &= \left\{ [A] \in \Gamma(\tilde{X}_{2n}) \mid \exists \tilde{f} \in Aff^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \text{ s.t. } D(\tilde{f}) = [A], \tilde{f}_*(\text{Gal}(\tilde{X}_{2n}/X)) = \right. \\ &\quad \left. \text{Gal}(\tilde{X}_{2n}/X) \right\}. \end{aligned}$$

To understand $\Gamma(X)$, we determine $\Gamma(\tilde{X}_{2n})$. The following theorem is a main theorem of this paper.

Theorem 4.2. For all $n \geq 4$, $\Gamma(\tilde{X}_{2n}) = \langle [R_{2n}], [T_{2n}] \rangle = \Gamma(P_{2n})$.

For the proof of theorem, we represent $A \in \text{SL}(2, \mathbb{R})$ by

$$A = \begin{pmatrix} r \cos \alpha(A) & s \cos \beta(A) \\ r \sin \alpha(A) & s \sin \beta(A) \end{pmatrix}$$

for some $r, s > 0$ and $\alpha(A), \beta(A)$ with $0 \leq \alpha(A) < \beta(A) < 2\pi$. And set $\theta(A) = \beta(A) - \alpha(A)$. This $\theta(A)$ means the angle of $A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The following two lemmas give the proof of the theorem.

Lemma 4.3. For $[A] \in \Gamma(\tilde{X}_{2n})$ with $|\cot \theta(A)| > \cot \frac{\pi}{2n}$, there exists $k, l \in \mathbb{Z}$ such that $|\cot \theta(A)| > |\cot \theta(T_{2n}^l R_{2n}^k A)|$.

Lemma 4.4. For $[A] \in \Gamma(\tilde{X}_{2n})$ with $|\cot \theta(A)| > \cot \frac{\pi}{2n}$, there exists $B \in \langle R_{2n}, T_{2n} \rangle$ such that $\cot \frac{\pi}{2n} \geq |\cot \theta(BA)|$.

PROOF OF THEOREM 4.2. $\Gamma(P_{2n}) \subseteq \Gamma(\tilde{X}_{2n})$ is clear since $\Gamma(\tilde{X}_{2n})$ is the universal Veech group of P_{2n} . We show $\Gamma(\tilde{X}_{2n}) \subseteq \Gamma(P_{2n})$. By Lemma 4.4, for each $[A] \in \Gamma(\tilde{X}_{2n})$, there exists $[B] \in \langle [R_{2n}], [T_{2n}] \rangle$ such that

$$\cot \frac{\pi}{2n} \geq |\cot \theta(BA)|.$$

If we map Q_1 of Figure 4 by an affine transformation BA , the image is parallelogram whose vertices correspond to vertices of $2n$ -gons and which has no such points in its interior. Moreover, it has the same area as Q_1 and each angle θ of its vertices satisfies $\pi/2n \leq \theta \leq \pi - \pi/2n$. We can see that such parallelograms are only

Q_1, Q_2, Q_3 and Q_4 of Figure 4 up to the image of them by $[R_{2n}]$ and $[T_{2n}]$. Then BA is either

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \cot \frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \cot \frac{\pi}{2n} \\ -\tan \frac{\pi}{2n} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \cot \frac{\pi}{2n} \\ -\tan \frac{\pi}{2n} & 0 \end{pmatrix}.$$

However, it does not happen except for the case that AB is the identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since every vertex of $2n$ -gons must be mapped to a vertex. Hence $BA = I$ and so $[A] = [B^{-1}] \in \langle [R_{2n}], [T_{2n}] \rangle = \Gamma(P_{2n})$. \square

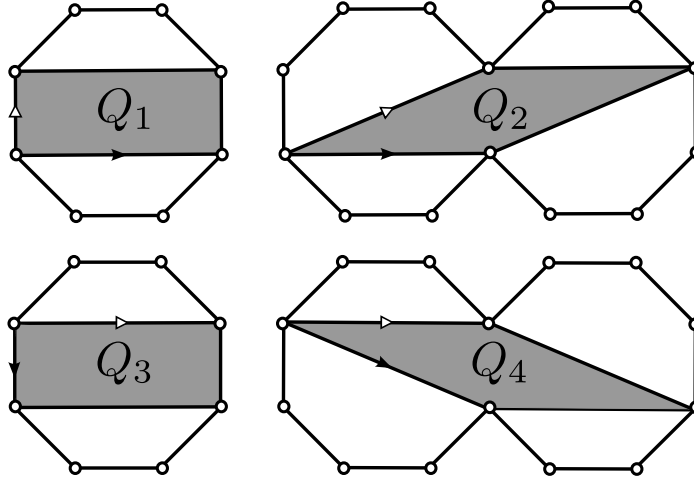


FIGURE 4.

PROOF OF LEMMA 4.3. We consider two cases : (a) $\cot \theta(A) > \cot \frac{\pi}{2n}$ and (b) $-\cot \theta(A) > \cot \frac{\pi}{2n}$

Case (a) : There exists $k \in \mathbb{Z}$ such that $B = R_{2n}^k A$ satisfies either $0 \leq \alpha(B) < \frac{\pi}{2n}$ or $\pi - \frac{\pi}{2n} \leq \alpha(B) < \pi$. We define the function

$$\begin{aligned} F_{\alpha(B)}^{\beta(B)}(x) = & 4 \cot^2 \frac{\pi}{2n} \cdot \frac{\sin \beta(B) \sin \alpha(B)}{\sin(\beta(B) - \alpha(B))} \cdot x^2 \\ & + 2 \cot \frac{\pi}{2n} \cdot \frac{\sin(\beta(B) + \alpha(B))}{\sin(\beta(B) - \alpha(B))} \cdot x + \cot \theta(B) \end{aligned}$$

of $x \in \mathbb{R}$. Note that $F_{\alpha(B)}^{\beta(B)}(l) = \cot(\beta(T_{2n}^l B) - \alpha(T_{2n}^l B)) = \cot \theta(T_{2n}^l B)$ for each $l \in \mathbb{Z}$.

(a)-1 : If $0 \leq \alpha(B) < \frac{\pi}{2n}$, there exists $\tilde{f} \in Aff^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ with $D(\tilde{f}) = [B]$. And \tilde{f} maps the rectangle Q_1 of Figure 4 to a parallelogram whose vertices correspond to vertices of $2n$ -gons and which has no such points in its interior. Hence we have $0 \leq \alpha(B) < \beta(B) \leq \frac{\pi}{2n}$. From this, if $\alpha(B) = 0$, then

$$F_0^{\beta(B)}(x) = 2 \cot \frac{\pi}{2n} \cdot x + \cot \theta(B)$$

and

$$F_0^{\beta(B)}(-\frac{1}{2}) = -\cot \frac{\pi}{2n} + \cot \theta(B) > 0.$$

So there exists a negative integer l such that

$$|F_0^{\beta(B)}(l)| < |F_0^{\beta(B)}(m)| \text{ for all } m \in \{0, -1, \dots, l+1, l-1\}.$$

Now we have

$$|\cot \theta(A)| = |F_0^{\beta(B)}(0)| > |F_0^{\beta(B)}(l)| = |\cot \theta(T_{2n}^l R_{2n}^k A)|.$$

If $0 < \alpha(B) < \beta(B) \leq \frac{\pi}{2n}$, then $F_{\alpha(B)}^{\beta(B)}(x)$ is a quadratic function of x and the axis of $F_{\alpha(B)}^{\beta(B)}$ is

$$x = -\frac{\cot \alpha(B) + \cot \beta(B)}{4 \cot \frac{\pi}{2n}} < -\frac{1}{2}$$

and

$$F_{\alpha(B)}^{\beta(B)}(-\frac{1}{2}) > 0.$$

Hence there exists a negative integer l such that

$$|F_{\alpha(B)}^{\beta(B)}(l)| < |F_{\alpha(B)}^{\beta(B)}(m)| \text{ for all } m \in \{0, -1, \dots, l+1, l-1\}.$$

And we have $|\cot \theta(A)| = |F_{\alpha(B)}^{\beta(B)}(0)| > |F_{\alpha(B)}^{\beta(B)}(l)| = |\cot \theta(T_{2n}^l R_{2n}^k A)|.$

(a)-2 : If $\pi - \frac{\pi}{2n} \leq \alpha(B) < \pi$, we have $\pi - \frac{\pi}{2n} \leq \alpha(B) < \beta(B) \leq \pi$ and

$$F_{\alpha(B)}^{\beta(B)}(x) = F_{(\pi-\beta(B))}^{(\pi-\alpha(B))}(-x).$$

By using the argument of (a)-1, we have $|\cot \theta(A)| > |\cot \theta(T_{2n}^l R_{2n}^k A)|$ for some $l \in \mathbb{Z}$.

Case (b) : We apply the same argument as in the Case (a) to the angle of two vectors $A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Then we have $|\cot \theta(A)| > |\cot \theta(T_{2n}^l R_{2n}^k A)|$ for some $k, l \in \mathbb{Z}$. \square

PROOF OF LEMMA 4.4. Let $[A]$ be an element in $\Gamma(\tilde{X}_{2n})$ with $|\cot \theta(A)| > \cot \frac{\pi}{2n}$. From the proof of Lemma 4.3, we obtain $A_1 = T_{2n}^{l_1} R_{2n}^{k_1} A$ with $|\cot \theta(A_1)| < |\cot \theta(A)|$ for some $k_1 \in \mathbb{Z}$ and $l_1 \in \mathbb{Z} - \{0\}$. If $|\cot \theta(A_1)| > \cot \frac{\pi}{2n}$, then we obtain $A_2 = T_{2n}^{l_2} R_{2n}^{k_2} A_1$ with $|\cot \theta(A_2)| < |\cot \theta(A_1)|$ for some $k_2, l_2 \in \mathbb{Z} - \{0\}$ from the proof of Lemma 4.3 again. We repeat this operation. If there exists $m_0 \in \mathbb{N}$ such that $\cot \frac{\pi}{2n} \geq |\cot \theta(A_{m_0})|$ holds, then $B = A_{m_0} A^{-1}$ is what we want. Suppose that $|\cot \theta(A_m)| > \cot \frac{\pi}{2n}$ holds for every $m \in \mathbb{N}$. Then we have an infinite sequence $\{A_m\}$ in $\langle R_{2n}, T_{2n} \rangle \cdot A$ with $|\cot \theta(A_{m-1})| > |\cot \theta(A_m)| > \cot \frac{\pi}{2n}$ for all m . We represent A_m by

$$A_m = \begin{pmatrix} r_m \cos \alpha_m & s_m \cos \beta_m \\ r_m \sin \alpha_m & s_m \sin \beta_m \end{pmatrix}$$

for some $r_m, s_m > 0$ and $0 \leq \alpha_m < \beta_m < 2\pi$ with $r_m s_m \sin(\beta_m - \alpha_m) = 1$. For each m , there exists $\tilde{f}_m \in Aff^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ such that $D(\tilde{f}_m) = [A_m]$ and \tilde{f}_m maps Euclidean segment which connect vertices of $2n$ -gons to other segment. Thus we have

$$r_m = \left| A_m \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| \geq 1 \text{ and } s_m = \left| A_m \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \geq 1.$$

Moreover, we have

$$r_m s_m = \frac{1}{\sin(\beta_m - \alpha_m)} \leq \frac{1}{\sin(\beta_1 - \alpha_1)}.$$

Hence $\{\alpha_m\}, \{\beta_m\}, \{r_m\}$ and $\{s_m\}$ are bounded and there exists a subsequence $\{A_{m_i}\}$ of $\{A_m\}$ such that A_{m_i} converges to some $A_\infty \in \text{SL}(2, \mathbb{R})$. Since $\{A_{m_i}\}$ is in a discrete set $\langle R_{2n}, T_{2n} \rangle \cdot A$, there exists $i_0 \in \mathbb{N}$ such that $A_{m_i} = A_\infty$ for all $i \geq i_0$. However, this contradicts the construction of the sequence $\{A_m\}$. Hence there exists $m_0 \in \mathbb{N}$ such that $\cot \frac{\pi}{2n} \geq |\cot \theta(A_{m_0})|$. \square

5. Calculation of Veech groups

Let X be an unramified finite covering of P_{2n} . By theorem 4.2, we can write $\Gamma(X)$ as follows.

$$\Gamma(X) = \left\{ [A] \in \langle [R_{2n}], [T_{2n}] \rangle \mid \exists \tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n}) \text{ s.t. } D(\tilde{f}) = [A], \right. \\ \left. \tilde{f}_*(\text{Gal}(\tilde{X}_{2n}/X)) = \text{Gal}(\tilde{X}_{2n}/X) \right\}.$$

Let z_0 be the point of P_{2n} which corresponds to the center of the $2n$ -gon Π_{2n} as in Example 3.2 and \bar{z}_0 be one of the preimages of z_0 in X . Let $\{x_1, x_2, \dots, x_n\}$ be the system of generators of $\pi_1(P_{2n}, z_0)$ as Figure 5. Then R_{2n} and T_{2n} define the following automorphisms $\gamma_{R_{2n}}$ and $\gamma_{T_{2n}}$ on $\pi_1(P_{2n}, z_0)$ (see Example 3.2).

$$\gamma_{R_{2n}} : \begin{cases} x_i \mapsto x_{i+1} & (i = 1, 2, \dots, n-1) \\ x_n \mapsto x_1^{-1} \end{cases}.$$

If n is even,

$$\gamma_{T_{2n}} : \begin{cases} x_1 \mapsto x_1 \\ x_{n+2-i}^{-1} x_i \mapsto x_{n+2-i}^{-1} x_i & (i = 2, 3, \dots, \frac{n}{2}) \\ x_i \mapsto (x_{n+2-i}^{-1} x_i) \cdots (x_{n-1}^{-1} x_3)(x_n^{-1} x_2) x_1^2 x_i & (i = 2, 3, \dots, \frac{n}{2}) \\ x_{\frac{n}{2}+1} \mapsto (x_{\frac{n}{2}+2}^{-1} x_{\frac{n}{2}}) \cdots (x_{n-1}^{-1} x_3)(x_n^{-1} x_2) x_1^2 x_{\frac{n}{2}+1} \end{cases}$$

and if n is odd,

$$\gamma_{T_{2n}} : \begin{cases} x_{n+1-i}^{-1} x_i \mapsto x_{n+1-i}^{-1} x_i & (i = 1, 2, \dots, \frac{n-1}{2}) \\ x_i \mapsto (x_{n+1-i}^{-1} x_i) \cdots (x_{n-1}^{-1} x_2)(x_n^{-1} x_1) x_i & (i = 1, 2, \dots, \frac{n-1}{2}) \\ x_{\frac{n+1}{2}} \mapsto (x_{\frac{n+3}{2}}^{-1} x_{\frac{n-1}{2}}) \cdots (x_{n-1}^{-1} x_2)(x_n^{-1} x_1) x_{\frac{n+1}{2}}. \end{cases}$$

Since $\text{Gal}(\tilde{X}_{2n}/P_{2n}) < \text{Ker}(D)$, $\text{Ker}(D)/\text{Gal}(\tilde{X}_{2n}/P_{2n}) = \{[id], [\tilde{h}^n]\}$ for some $\tilde{h} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ with $D(\tilde{h}) = [R_{2n}]$ and each element in $\text{Gal}(\tilde{X}_{2n}/P_{2n})$ defines an inner automorphism of $\text{Gal}(\tilde{X}_{2n}/P_{2n}) \cong \pi_1(P_{2n}, z_0)$, the action of each element of $\text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$ on $\pi_1(P_{2n}, z_0)$ can be represented by a composition of $\gamma_{R_{2n}}$, $\gamma_{T_{2n}}$ and inner automorphisms of $\pi_1(P_{2n}, z_0)$.

Hence we have the following.

Proposition 5.1. *For $\tilde{f} \in \text{Aff}^+(\tilde{X}_{2n}, \tilde{\varphi}_{2n})$, following two are equivalent. Here A is one of elements in $D(\tilde{f})$.*

- The mapping \tilde{f} satisfies $\tilde{f}_*(\text{Gal}(\tilde{X}_{2n}/X)) = \text{Gal}(\tilde{X}_{2n}/X)$.
- There exists one of the preimages $\bar{z}_1 \in X$ of z_0 such that

$$\gamma_A(\pi_1(X, \bar{z}_0)) = \pi_1(X, \bar{z}_1) \text{ or } \gamma_{-A}(\pi_1(X, \bar{z}_0)) = \pi_1(X, \bar{z}_1).$$

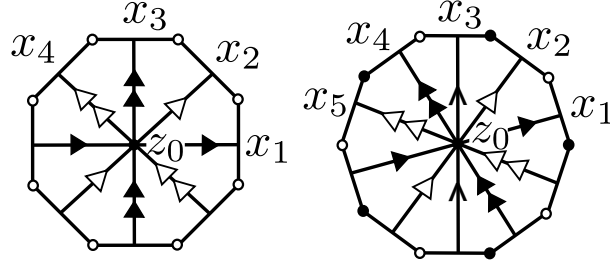
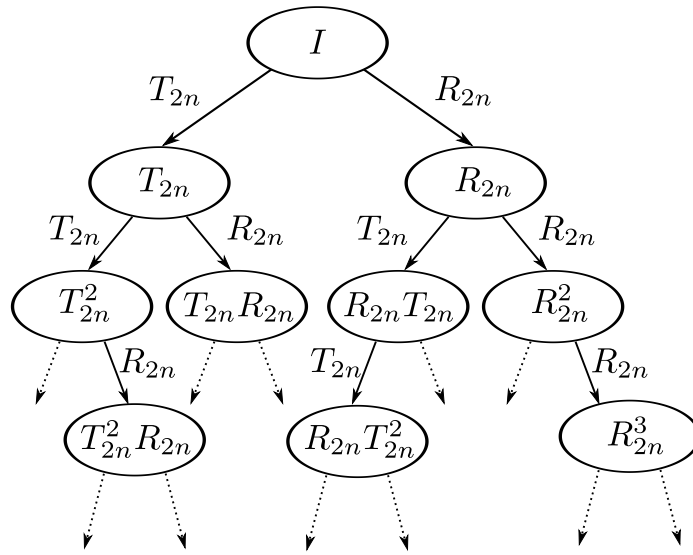


FIGURE 5.

By using this condition, we can determine whether $[A]$ is in $\Gamma(X)$ or not for each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$.

Now we can calculate the Veech group $\Gamma(X)$ of an unramified finite covering X of P_{2n} by using the following method. Schmithüsen ([Sch04]) also use this method to the calculations of Veech groups of origamis. The calculation is done on the following tree which we explain below.



Calculation of $\Gamma(X)$ (Reidemeister-Schreier method).

Given an unramified finite covering X of P_{2n} .
 Let **Rep** and **Gen** be empty sets.
 Add $[I]$ to **Rep**. Set $A = I$.
 Loop:
 Set $B = A \cdot T_{2n}$, $C = A \cdot R_{2n}$.
 Check whether B is already represented by **Rep**:
 For each $[D]$ in **Rep**, check whether $[B] \cdot [D]^{-1}$ is in $\Gamma(X)$.
 If so, add $[B] \cdot [D]^{-1}$ to **Gen**.
 If none is found, add $[B]$ to **Rep**.
 Do the same for C instead of B .
 If there exists a successor of A in **Rep**,
 let A be this successor and go to the beginning of the loop.
 If not, finish the loop.

Result:
Gen : a list of generators of $\Gamma(X)$.
Rep : a list of coset representatives in $\langle [R_{2n}], [T_{2n}] \rangle$.

Proposition 5.2. *Let X be an unramified finite covering of P_{2n} . Then we have the following properties.*

- (1) *Any two elements in **Rep** belong to different cosets.*
- (2) *The calculation stops in finitely many steps.*
- (3) *In the end, each coset is represented by a member of **Rep**.*
- (4) *In the end, $\Gamma(X)$ is generated by the elements in **Gen**.*

PROOF. (1) is clear and we can see a proof of (3), (4) in [Sch04]. (2) is equivalent to what $\Gamma(X)$ is a finite index subgroup of $\langle [R_{2n}], [T_{2n}] \rangle$. By the next proposition, we conclude that $\Gamma(X)$ and $\Gamma(P_{2n}) = \langle [R_{2n}], [T_{2n}] \rangle$ are commensurable. Hence $\Gamma(X)$ is a finite index subgroup of $\langle [R_{2n}], [T_{2n}] \rangle$. Since all elements in **Rep** belong to different cosets of $\Gamma(X)$ in $\langle [R_{2n}], [T_{2n}] \rangle$, $\#\mathbf{Rep}$ cannot be greater than this index and hence the calculation of $\Gamma(X)$ stops in finitely many steps. \square

For a Riemann surface X and a holomorphic quadratic differential φ , denote by $C(X, \varphi)$ the set of all zeros of φ and punctures of X .

Proposition 5.3. ([GJ96] and [GJ00].) *Let $p : X \rightarrow Y$ be a covering mapping between Riemann surfaces. Let φ_X be a holomorphic quadratic differential on X and set $\varphi_Y = p_*\varphi_X$. Suppose that $p(C(Y, \varphi_Y)) = C(X, \varphi_X)$ and $p^{-1}(C(X, \varphi_X)) = C(Y, \varphi_Y)$. Then the Veech groups $\Gamma(X, \varphi_X)$ and $\Gamma(Y, \varphi_Y)$ are commensurable.*

Example 5.4. Let X be the covering of P_8 as Figure 6. We calculate the Veech group $\Gamma(X)$.

The fundamental group of X is

$$\pi_1(X, \bar{z}_0) = \langle x_1^2, x_2, x_4, x_1x_3, x_3x_1, x_1^{-1}x_2x_1, x_1^{-1}x_4x_1 \rangle.$$

Loop 1 : **Rep** = $\{[I]\}$, **Gen** = \emptyset , $A = I$, $B = T_8$, $C = R_8$.

We check $[B] \cdot [I]^{-1} = [T_8]$; the homomorphism γ_{T_8} maps the generators

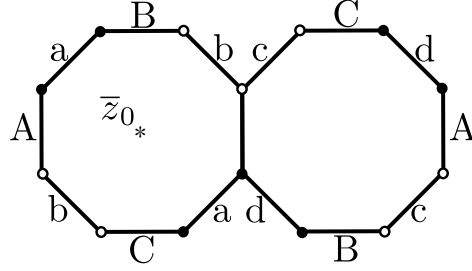


FIGURE 6.

of $\pi_1(X, \bar{z}_0)$ as follows

$$\gamma_{T_8} : \begin{cases} x_1^2 \mapsto x_1^2 \\ x_2 \mapsto x_4^{-1} x_2 x_1^2 x_2 \\ x_4 \mapsto x_4^{-1} x_2 x_1^2 x_4 \\ x_1 x_3 \mapsto x_1 x_4^{-1} x_2 x_1^2 x_3 \\ x_3 x_1 \mapsto x_4^{-1} x_2 x_1^2 x_3 x_1 \\ x_1^{-1} x_2 x_1 \mapsto x_1^{-1} x_4^{-1} x_2 x_1^2 x_2 x_1 \\ x_1^{-1} x_4 x_1 \mapsto x_1^{-1} x_4^{-1} x_2 x_1^2 x_4 x_1 \end{cases}.$$

By taking \bar{z}_0 as a base point, all images represent closed curves. Hence $[T_8]$ is an element in $\Gamma(X)$ and add $[T_8]$ in **Gen**.

Now **Rep** = $\{[I]\}$, **Gen** = $\{[T_8]\}$.

We check $[C] \cdot [I]^{-1} = [R_8]$; there is no point of X such that $\gamma_{R_{2n}}(x_2) = x_3$ or $\gamma_{-R_{2n}}(x_2) = x_3^{-1}$ represent closed curves with the point as a base point. Hence $[R_8]$ is not in $\Gamma(X)$. We add $[R_8]$ in **Rep**.

Now **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8]\}$ and R_8 is a successor of $A = I$ and is in **Rep**. We set $A = R_8$.

Loop 2 : **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8]\}$, $A = R_8$, $B = R_8 T_8$, $C = R_8^2$.

We check $[B] \cdot [I]^{-1} = [R_8 T_8]$; it is not in $\Gamma(X)$.

We check $[B] \cdot [R_8]^{-1} = [R_8 T_8 R_8^{-1}]$; the homomorphism $\gamma_{R_8 T_8 R_8^{-1}}$ is the form

$$\gamma_{R_8 T_8 R_8^{-1}} : \begin{cases} x_1 \mapsto x_1 x_2^{-2} x_3^{-1} x_1^{-1} \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_1 x_3 x_2^2 x_3 \\ x_4 \mapsto x_1 x_3 x_2^2 x_4 \end{cases}$$

and maps the generators of $\pi_1(X, \bar{z}_0)$ as follows

$$\gamma_{R_8 T_8 R_8^{-1}} : \begin{cases} x_1^2 \mapsto x_1 x_2^{-2} x_3^{-1} x_2^{-2} x_3^{-1} x_1^{-1} \\ x_2 \mapsto x_2 \\ x_4 \mapsto x_1 x_3 x_2^2 x_4 \\ x_1 x_3 \mapsto x_1 x_3 \\ x_3 x_1 \mapsto x_1 x_3 x_2^2 x_3 x_1 x_2^{-2} x_3^{-1} x_1^{-1} \\ x_1^{-1} x_2 x_1 \mapsto x_1 x_3 x_2^2 x_1^{-1} x_2 x_1 x_2^{-2} x_3^{-1} x_1^{-1} \\ x_1^{-1} x_4 x_1 \mapsto x_1 x_3 x_2^2 x_3 x_2^2 x_4 x_1 x_2^{-2} x_3^{-1} x_1^{-1} \end{cases}.$$

By taking \bar{z}_0 as a base point, all images represent closed curves. Hence $[R_8 T_8 R_8^{-1}]$ is an element in $\Gamma(X)$ and add $[R_8 T_8 R_8^{-1}]$ in **Gen**.

Now **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8], [R_8 T_8 R_8^{-1}]\}$.
We check $[C] \cdot [I]^{-1} = [R_8^2]$; the homomorphism γ_{R_8} maps the generators of $\pi_1(X, \bar{z}_0)$ as follows

$$\gamma_{R_8^2} : \begin{cases} x_1^2 \mapsto x_3^2 \\ x_2 \mapsto x_4 \\ x_4 \mapsto x_2^{-1} \\ x_1 x_3 \mapsto x_3 x_1^{-1} \\ x_3 x_1 \mapsto x_1^{-1} x_3 \\ x_1^{-1} x_2 x_1 \mapsto x_3^{-1} x_4 x_3 \\ x_1^{-1} x_4 x_1 \mapsto x_3^{-1} x_2^{-1} x_3 \end{cases}.$$

By taking \bar{z}_0 as a base point, all images represent closed curves. Hence $[R_8^2]$ is an element in $\Gamma(X)$ and add $[R_8^2]$ in **Gen**.

We check $[C] \cdot [R_8]^{-1} = [R_8]$; it is not in $\Gamma(X)$.

Now, **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8], [R_8 T_8 R_8^{-1}], [R_8^2]\}$ and there is no successor of $A = R_8$ in **Rep**. We finish the loop.

Result : **Rep** = $\{[I], [R_8]\}$, **Gen** = $\{[T_8], [R_8 T_8 R_8^{-1}], [R_8^2]\}$.

As a result, $\Gamma(X) = \langle [T_8], [R_8 T_8 R_8^{-1}], [R_8^2] \rangle$ and coset representatives in $\langle [R_8], [T_8] \rangle$ is $\{[I], [R_8]\}$.

Remark. In the case of origamis, Schmithüsen showed that the calculations always stop by connecting the Veech groups of origami with subgroups of $\mathrm{SL}(2, \mathbb{Z})$ (see [Sch04]). In our case, for certain Abelian coverings of $2n$ -gons, we connect the Veech groups with subgroups of $\mathrm{SL}(n, \mathbb{Z}_d)$ and calculate the Veech groups by using the corresponding matrices. It is seen in section 7.

6. Calculation of $\mathbb{H}/\Gamma(X)$

Let X be an unramified finite covering of P_{2n} . Assume that the calculation of $\Gamma(X)$ by the Reidemeister-Schreier method stopped. Then **Gen** is a list of generators of $\Gamma(X)$ and **Rep** is a list of coset representatives in $\langle [R_{2n}], [T_{2n}] \rangle$.

Let D be the fundamental domain of $\langle [R_{2n}], [T_{2n}] \rangle$ in \mathbb{H} as Figure 7. Then

$$F = \mathrm{Int} \left(\bigcup_{[A] \in \mathbf{Rep}} [A](\bar{D}) \right)$$

is a fundamental domain of $\Gamma(X)$. Here $[A]$ means a Möbius transformation.

By reading **Gen**, we can know all pairs of sides of F which are identified by the action of $\Gamma(X)$. We call sides of $[A](D)$ which correspond to $(-\cot \frac{\pi}{2n}, i)$, $(\cot \frac{\pi}{2n}, i)$, $(-\cot \frac{\pi}{2n}, i\infty)$ and $(\cot \frac{\pi}{2n}, i\infty)$ the R^{-1} -side, the R -side, the T^{-1} -side and the T -side of $[A]$, respectively.

Proposition 6.1. Assume that **Rep** = $\{[A_1], [A_2], \dots, [A_k]\}$. Then for each $i, j \in \{1, 2, \dots, k\}$,

- The T -side of $[A_j]$ and the T^{-1} -side of $[A_i]$ are identified if and only if $[A_j T_{2n} A_i^{-1}] \in \Gamma(X)$.

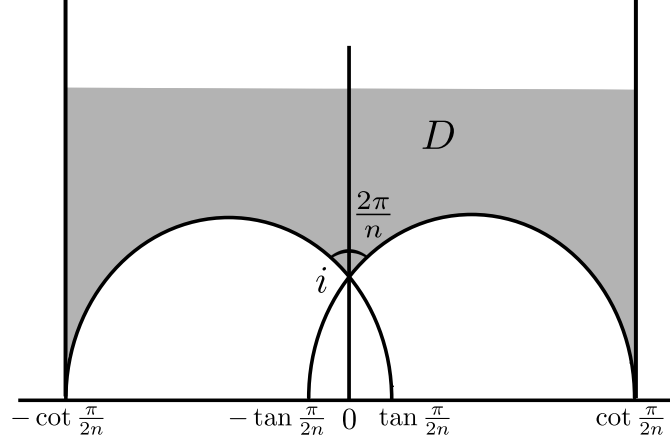


FIGURE 7.

- The R -side of $[A_j]$ and the R^{-1} -side of $[A_i]$ are identified if and only if $[A_j R_{2n} A_i^{-1}] \in \Gamma(X)$.

We give a triangulation of $\mathbb{H}/\Gamma(X)$ by decomposing D as Figure 8. Then the number of triangles and sides are $2 \cdot \#\mathbf{Rep}$ and $3 \cdot \#\mathbf{Rep}$, respectively. Moreover, we can calculate the number v of vertices by using Proposition 6.1. When we calculate v , we decompose v as $v = v_\infty + v_{\cot} + v_{\text{cone}}$. Here v_∞ is the number of vertices corresponding to ∞ of D , v_{\cot} is the number of vertices corresponding to $\pm \cot \frac{\pi}{2n}$ of D and v_{cone} is the number of vertices corresponding to i of D . Then $\mathbb{H}/\Gamma(X)$ has genus $(2 + \#\mathbf{Rep} - v)/2$ and $v_\infty + v_{\cot}$ punctures. We can also calculate the number of cone points and their orders in the calculation of v_{cone} .

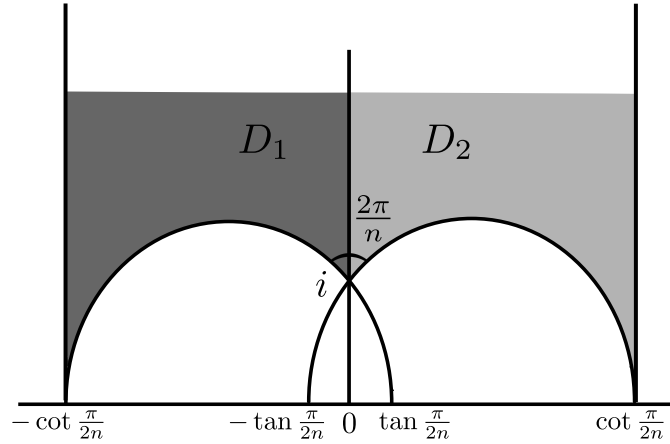


FIGURE 8.

Example 6.2. Let X be the Riemann surface as Figure 6. At the end of the calculation of $\Gamma(X)$, we have $\mathbf{Gen} = \{[T_8], [R_8 T_8 R_8^{-1}], [R_8^2]\}$ and $\mathbf{Rep} = \{[I], [R_8]\}$.

FIGURE 9.

7. Veech groups of Abelian coverings

In this section, we show that the calculation of Veech group $\Gamma(X)$ by the Reidemeister-Schreier method always stops if X is a finite Abelian covering of P_{2n} . And we show that the calculations of Veech groups of certain Abelian coverings can be done by using the corresponding subgroups of $\mathrm{SL}(n, \mathbb{Z}_d)$.

Recall that if $\Gamma(X)$ is a finite index subgroup of $\langle [R_{2n}], [T_{2n}] \rangle$, then the calculation of $\Gamma(X)$ stops by the proof of Proposition 5.2. We have a partial answer about the stop of calculations.

Theorem 7.1. *Let X be a finite Abelian covering of P_{2n} , that is, X is a finite Galois covering of P_{2n} and $\text{Gal}(X/P_{2n})$ is an Abelian group. Then the calculation of $\Gamma(X)$ stops.*

PROOF. Recall that z_0 is the point of P_{2n} which corresponds to the center of the $2n$ -gon Π_{2n} as in Example 3.2 and $\bar{z}_0 \in X$ is one of the preimages of z_0 . Since X is a Galois covering, for each $w \in \text{Gal}(\tilde{X}_{2n}/P_{2n}) = \langle x_1, x_2, \dots, x_n \rangle$, w is in $\pi_1(X, \bar{z}_0)$ if and only if w is in $\pi_1(X, \bar{z}_1)$ for all $\bar{z}_1 \in X$. Hence $[A]$ is in $\Gamma(X)$ if and only if γ_A or γ_{-A} fix $\pi_1(X, \bar{z}_0)$ for each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$.

As $\text{Gal}(X/P_{2n}) \cong \pi_1(P_{2n}, z_0)/\pi_1(X, \bar{z}_0)$ is an Abelian group, $\overline{x_i x_j} = \overline{x_j x_i}$ and $x_i x_j = x_j x_i \cdot w$ for some $w \in \pi_1(X, \bar{z}_0)$. Moreover, set $d = \text{lcm}\{\text{ord}(\bar{\tau}_1), \text{ord}(\bar{\tau}_2), \dots, \text{ord}(\bar{\tau}_n)\}$, then $x_i^d \in \pi_1(X, \bar{z}_1)$ for all i and all $\bar{z}_1 \in X$.

Set $(e_1, e_2, \dots, e_{2n}) = I_{2n}$. We consider the homomorphism $\nu : \text{Gal}(\tilde{X}_{2n}/P_{2n}) \rightarrow \mathbb{Z}_d^n$; $x_i \mapsto e_i$. Then there exists a homomorphism $\Phi_d : \langle \gamma_T, \gamma_R \rangle \rightarrow \text{SL}(n, \mathbb{Z}_d)$ such

that the following diagram is commutative.

$$\begin{array}{ccc} \text{Gal}(\tilde{X}_{2n}/P_{2n}) & \xrightarrow{\gamma_A} & \text{Gal}(\tilde{X}_{2n}/P_{2n}) \\ \nu \downarrow & & \downarrow \nu \\ \mathbb{Z}_d^n & \xrightarrow{\Phi_d(A)} & \mathbb{Z}_d^n \end{array}$$

Set $V = \nu(\pi_1(X, \bar{z}_0))$. For each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$, if $[A]$ satisfies $\Phi_d(A)(V) = V$, then $[A] \in \Gamma(X)$.

Now we conclude that $\sharp \mathbf{Rep} \leq \sharp \text{SL}(n, \mathbb{Z}_d)$ at every step of the calculation. Suppose that $\sharp \mathbf{Rep} > \sharp \text{SL}(n, \mathbb{Z}_d)$ happens at some step of the calculation. Then there exists two distinct elements $[A]$ and $[B]$ in \mathbf{Rep} such that $\Phi_d(A) = \Phi_d(B)$. Since $\Phi_d(AB^{-1}) = I_n$ stabilizes V , $[A] \cdot [B]^{-1}$ is in $\Gamma(X)$. However, since $[A]$ and $[B]$ are distinct elements in \mathbf{Rep} , $[A] \cdot [B]^{-1}$ is not in $\Gamma(X)$. This is a contradiction. \square

From the proof of theorem 7.1, we have the following.

Corollary 7.2. *Let X be a finite Abelian covering of P_{2n} . If there exists $d \in \mathbb{N}$ such that $\{\text{ord}(\bar{x}_1), \text{ord}(\bar{x}_2), \dots, \text{ord}(\bar{x}_n)\} = \{d\}$ or $\{1, d\}$, then $[A] \in \Gamma(X)$ if and only if $\Phi_d(A)(V) = V$ for each $[A] \in \langle [R_{2n}], [T_{2n}] \rangle$.*

Example 7.3. Let X be the covering of P_8 the same as Figure 6. Then X satisfies the assumption of Corollary 7.2. The fundamental group of X is

$$\pi_1(X, \bar{z}_0) = \langle x_1^2, x_2, x_4, x_1x_3, x_3x_1, x_1^{-1}x_2x_1, x_1^{-1}x_4x_1 \rangle$$

and

$$V = \langle e_2, e_4, e_1 + e_3 \rangle_{\mathbb{Z}_2}.$$

By Corollary 7.2, for $[A] \in \langle [R_8], [T_8] \rangle$, $[A]$ is in $\Gamma(X)$ if and only if $\Phi_2(A)$ satisfies the followings :

$$\begin{cases} \Phi_2(A)_{1,1} + \Phi_2(A)_{3,1} + \Phi_2(A)_{1,3} + \Phi_2(A)_{3,3} \equiv 0 \pmod{2} \\ \Phi_2(A)_{1,j} + \Phi_2(A)_{3,j} \equiv 0 \pmod{2} (j = 2, 4). \end{cases}$$

8. Examples.

Finally we show some examples of Veech groups that are calculated by the method of this paper.

Example 8.1. Let X be the double covering of P_8 as Figure 10. Then X is a Riemann surface of type $(3, 2)$. Set $R = [R_8]$, $T = [T_8]$. Then

- For $[A] \in \langle R, T \rangle$, $[A]$ is in $\Gamma(X)$ if and only if $\Phi_2(A)_{1,j} \equiv 0 \pmod{2}$ ($j = 2, 3, 4$),
- $\Gamma(X)$

$$= \left\langle T, RT^2R^{-1}, RTRT^2(RTR)^{-1}, (RT)^3(RTRTR)^{-1}, \right.$$

$$(RT)^2R^2T(RTR^2)^{-1}, (RT)^2R^3T(RTRTR^3)^{-1},$$

$$RTR^2T(RTRTR^2)^{-1}, RTR^3T^2(RTR^3)^{-1},$$

$$\left. RTR^3TR, R^2TR^{-2}, R^3(RTR^3T)^{-1} \right\rangle,$$
- $\Gamma(X) \setminus \langle R, T \rangle$

$$= \left\{ \begin{array}{l} I, R, RT, R^2, RTR, RTRT, RTR^2, RTRTR, \\ RTRTR^2, RTRTR^3, RTR^3, RTR^3T \end{array} \right\} \text{ and}$$

- $\mathbb{H}/\Gamma(X)$ is a Riemann surface of type $(0, 11)$.

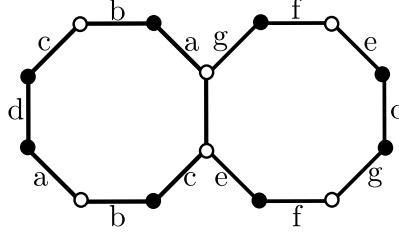


FIGURE 10.

Example 8.2. Let X be the covering of P_8 as Figure 11. Then X is a Riemann surface of type $(5, 4)$. Set $R = [R_8]$, $T = [T_8]$. Then

- For $[A] \in \langle R, T \rangle$, $[A]$ is in $\Gamma(X)$ if and only if $\Phi_4(A)$ satisfies the followings
:

$$\left\{ \begin{array}{l} \sum_{i=1}^2 (\Phi_4(A)_{i,2} - \Phi_4(A)_{i,1}) \equiv \sum_{i=3}^4 (\Phi_4(A)_{i,2} - \Phi_4(A)_{i,1}) \pmod{4}, \\ \sum_{i=1}^2 (\Phi_4(A)_{i,1} + \Phi_4(A)_{i,j}) \equiv \sum_{i=3}^4 (\Phi_4(A)_{i,1} + \Phi_4(A)_{i,j}) \pmod{4} (j = 3, 4), \end{array} \right.$$

- $\Gamma(X) = \langle T, R^2(RT)^{-1}, RT^2R^{-1}, RTRT(RTR)^{-1}, RTR^2 \rangle$,
- $\Gamma(X) \setminus \langle R, T \rangle = \{I, R, RT, RTR\}$ and
- $\mathbb{H}/\Gamma(X)$ is a Riemann surface of type $(0, 5)$.

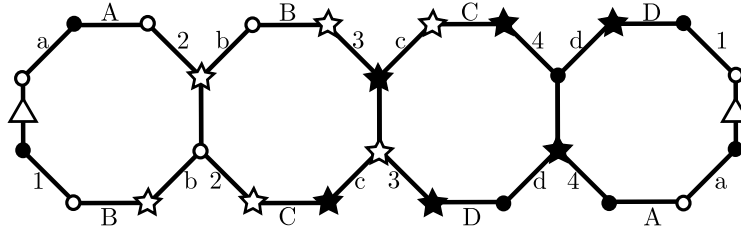


FIGURE 11.

Example 8.3. $n \geq 2$. Let X_{4n} be the double covering of P_{4n} as Figure 12. That is, X_{4n} is constructed by gluing two regular $4n$ -gons. Labels of small and capital letters appear in turn. The sides whose labels are capital letters are identified with the opposite sides of another polygon and others are identified with the opposite sides of the same polygon. Then X_{4n} is a Riemann surface of type $(2n - 1, 2)$.

- For $[A] \in \langle [R_{4n}], [T_{4n}] \rangle$, $[A]$ is in $\Gamma(X_{4n})$ if and only if $\Phi_2(A)$ satisfies the followings :

$$\begin{cases} \sum_{i=1}^n \Phi_2(A)_{2i-1,2j} \equiv 0 \pmod{2} (j = 1, \dots, n), \\ \sum_{i=1}^n (\Phi_2(A)_{2i-1,1} + \Phi_2(A)_{2i-1,2j-1}) \equiv 0 \pmod{2} (j = 2, \dots, n), \end{cases}$$

- $\Gamma(X_{4n}) = \langle [T_{4n}], [R_{4n}T_{4n}R_{4n}^{-1}], [R_{4n}^2] \rangle$,
- $\Gamma(X_{4n}) \setminus \langle [R_{4n}], [T_{4n}] \rangle = \{[I], [R_{4n}]\}$ and
- $\mathbb{H}/\Gamma(X_{4n})$ is an orbifold which has no genus, 3 punctures and one cone point whose order is n .

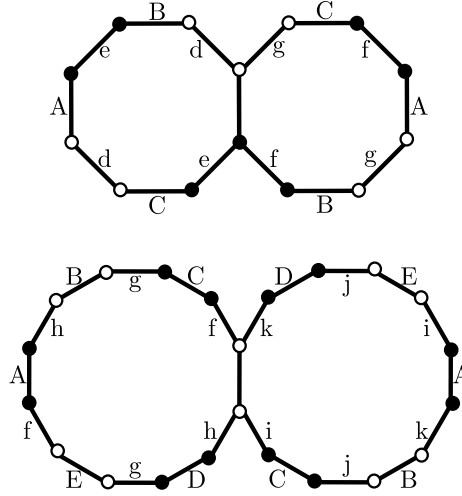


FIGURE 12.

Example 8.4. For each $n \geq 2$, let X_{4n} be the double covering of P_{4n} as Figure 13. That is, horizontal and vertical sides of two polygons are identified with the opposite sides of another polygon and others are identified with the opposite sides of the same polygon. Then X_{4n} is a Riemann surface of type $(2n - 1, 2)$.

- For $[A] \in \langle [R_{4n}], [T_{4n}] \rangle$, $[A]$ is in $\Gamma(X_{4n})$ if and only if $\Phi_2(A)$ satisfies the followings :

$$\begin{cases} \Phi_2(A)_{1,1} + \Phi_2(A)_{n+1,1} + \Phi_2(A)_{n+1,n+1} + \Phi_2(A)_{n+1,n+1} \equiv 0 \pmod{2}, \\ \Phi_2(A)_{1,j} + \Phi_2(A)_{n+1,j} \equiv 0 \pmod{2} (j = 2, \dots, n, n+2, \dots, 2n), \end{cases}$$

- $\Gamma(X_{4n}) = \langle [R_{4n}^i T_{4n} R_{4n}^{-i}], [R_{4n}^n] \mid i = 0, 1, \dots, n-1 \rangle$,
- $\Gamma(X_{4n}) \setminus \langle [R_{4n}], [T_{4n}] \rangle = \{[I], [R_{4n}], [R_{4n}^2], \dots, [R_{4n}^{n-1}]\}$ and
- $\mathbb{H}/\Gamma(X_{4n})$ is an orbifold which has no genus, $2n + 1$ punctures and one cone point whose order is 2.

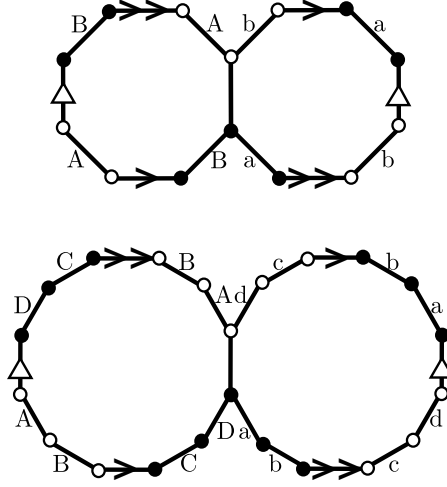


FIGURE 13.

Example 8.5. Let X_d be the covering of P_8 with degree d as Figure 14. Then X_d is a Riemann surface of type $(d+1, d)$. And, for $[A] \in \langle [R_8], [T_8] \rangle$, $[A]$ is in $\Gamma(X_d)$ if and only if $\Phi_d(A)_{1,j} \equiv 0 \pmod{d} (j = 2, 3, 4)$. The next is a chart about Veech groups $\Gamma(X_d)$. Here,

- $\# \text{ Rep}$ is the index of $\Gamma(X_d)$ in $\langle [R_8], [T_8] \rangle$,
- $\# \text{ Gen}$ is a number of generators of $\Gamma(X_d)$ by this calculation,
- “genus” is the genus of $\mathbb{H}/\Gamma(X_d)$,
- “puncture” is the number of punctures of $\mathbb{H}/\Gamma(X_d)$ and
- “cone point (order)” is the number of cone points of $\mathbb{H}/\Gamma(X_d)$ and their orders.

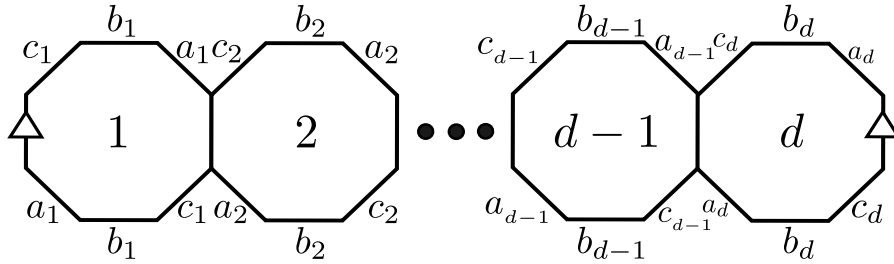


FIGURE 14.

d	# Gen	# Rep	genus	puncture	cone point (order)
2	11	12	0	11	0
3	29	32	1	24	0
4	87	96	8	58	0
5	142	156	24	68	6 (2,2,2,2,2,2)
6	349	384	45	200	0
7	367	400	87	128	0
8	704	768	149	280	0
9	785	864	185	280	0
10	1704	1872	419	568	0
11	1353	1464	400	300	0

Example 8.6. Let X_d be the covering of P_8 with degree d as Figure 15. Then X_d is a Riemann surface of type $(d+1, d)$. And, for $[A] \in \langle [R_8], [T_8] \rangle$, $[A]$ is in $\Gamma(X_d)$ if and only if $\sum_{i=1}^4 (\Phi_d(A)_{i,j} - \Phi_d(A)_{i,1}) \equiv 0 \pmod{d} (j = 2, 3, 4)$. The next is a chart about Veech groups $\Gamma(X_d)$.

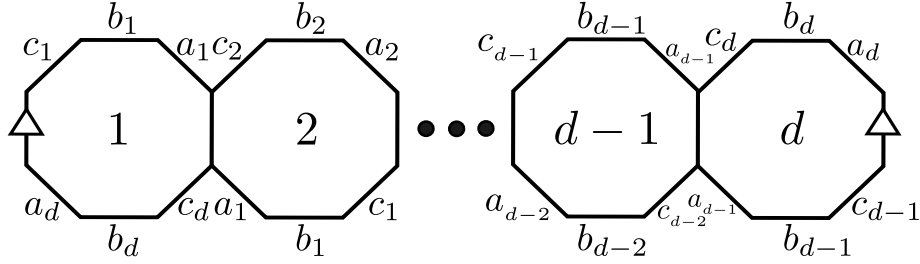


FIGURE 15.

d	# Gen	# Rep	genus	puncture	cone point (order)
2	2	1	0	2	1 (4)
3	29	32	1	24	0
4	5	4	0	5	0
5	142	156	24	68	6 (2,2,2,2,2,2)
6	29	32	1	24	0
7	367	400	87	128	0
8	29	32	1	24	0
9	789	864	185	280	0
10	142	156	24	68	6 (2,2,2,2,2,2)
11	1353	1464	400	300	0
12	115	128	11	76	0
13	2220	2380	682	416	14 (2,2,2,2,2,2,2,2,2,2,2,2,2,2)
14	367	400	87	128	0

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References

- [EG97] C. Earle and F. Gardiner. Teichmüller disks and Veech’s F-structures. *Contemporary Mathematics*, 201:165–189, 1997.
- [GJ96] E. Gutkin and C. Judge. The geometry and arithmetic of translation surfaces with applications to polygonal billiards. *Math. Res. Lett.*, 3(3):391–403, 1996.
- [GJ00] E. Gutkin and C. Judge. Affine mappings of translation surfaces: geometry and arithmetic. *Duke Math. J.*, 103(2):191–213, 2000.
- [Sch04] G. Schmithüsen. An algorithm for finding the Veech group of an Origami. *Experimental Mathematics*, 13:459–472, 2004.
- [Sin72] D. Singerman. Finitely maximal Fuchsian groups. *J. London Math. Soc.*, 6:29–38, 1972.
- [Vee89] W. Veech. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Inventiones Mathematicae*, 97(3):553–583, 1989.
- [Vee91] W. Veech. Erratum: Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Inventiones Mathematicae*, 103(2):447, 1991.

DEPARTMENT OF MATHEMATICS TOKYO INSTITUTE OF TECHNOLOGY 2-12-1 OOKAYAMA, MEGURO-KU, TOKYO 152-8551, JAPAN

E-mail address: `shinomiya.y.aa@m.titech.ac.jp`